Curl, divergence and whatever's left over

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As always, we refer to the example of fluids. Consider the velocity field $\mathbf{u}(\mathbf{r})$, and we will stack up the gradients for each component of velocity $\nabla u_x \cdot \delta\mathbf{r} \approx \delta u_x$ into a matrix G:

$$G = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{1}$$

from which we can write sensible-looking statements such as $\delta\mathbf{u} \approx G\delta\mathbf{r}$. We are simple folk, so we manipulate G into a more interesting form by:

$$\begin{aligned} G &= \frac{1}{2}(G + G) && \text{(Multiplication by 1)} \\ G &= \frac{1}{2}(G + G^\mathsf{T} + G - G^\mathsf{T}) && \text{(Addition of 0)} \\ G &= \frac{1}{2}(G + G^\mathsf{T}) + \frac{1}{2}(G - G^\mathsf{T}) \\ G &= D + W \end{aligned} \tag{2}$$

where we have a symmetric matrix D and an antisymmetric matrix W. Let's look at W first:

$$W = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} & 0 & \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z} & \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} & 0 \end{bmatrix} \tag{3}$$

which already looks awfully familiar. Taking the Cartesian definition of the curl, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, we find:

$$W = \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \frac{1}{2}[\boldsymbol{\omega}]_\times \tag{4}$$

and this final notation expresses the fact that this is the 'cross-product matrix' form of $\boldsymbol{\omega}$, i.e.

$$W\mathbf{v} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{v}. \tag{5}$$

We are going to consider a little piece of fluid — the reader is free to choose whether they prefer to think in terms of cubes, spheres or potatoes: the results hold regardless. For a point within the fluid around $\mathbf{r}_0$:

$$\begin{aligned} \mathbf{u}(\mathbf{r}_0 + \delta\mathbf{r}) &\approx \mathbf{u}(\mathbf{r}_0) + G(\mathbf{r}_0)\delta\mathbf{r} \\ &\approx \mathbf{u}(\mathbf{r}_0) + D(\mathbf{r}_0)\delta\mathbf{r} + W(\mathbf{r}_0)\delta\mathbf{r} \\ &\approx \mathbf{u}(\mathbf{r}_0) + D(\mathbf{r}_0)\delta\mathbf{r} + \frac{1}{2}\boldsymbol{\omega} \times \delta\mathbf{r} \end{aligned} \tag{6}$$

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We already see the relationship between curl and local rigid-body rotation (and why factors of two start springing up). Let's turn our attention to D:

$$D = \frac{1}{2}\begin{bmatrix} 2\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & 2\frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} & \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} & 2\frac{\partial u_z}{\partial z} \end{bmatrix} \tag{7}$$

Being symmetric, we know that it has real eigenvalues and orthogonal eigenvectors, which represents the stretching of a sphere into an ellipsoid (and therefore contains the information about the deformation of our fluid chunk). You may have encountered the trace of a matrix before (the sum of the diagonal):

$$\begin{aligned} \mathrm{tr}(D) &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{u} \\ &= \lambda_1 + \lambda_2 + \lambda_3 \end{aligned} \tag{8}$$

where the λ are the eigenvalues of D. Supposing that our fluid is incompressible, $\nabla \cdot \mathbf{u} = 0$, and so the sum of the eigenvalues of D must be zero. Is this just serendipity, or can we pull a physical interpretation out of this?

Considering the chunk of fluid deforming, each particle moves to a new position $\mathbf{r}(t_0 + \delta t) \approx \mathbf{r}(t_0) + \mathbf{u}\delta t$. Defining the chunk mass $m(t)$ by the integral

$$m(t) = \int_{\mathbf{r} \in R(t)} \rho(\mathbf{r}) dV \tag{9}$$

where $R(t)$ is the set of points defined by the pointwise evolution of the set $R_0 = R(t_0)$. As long as we define our R_0 nicely enough (e.g. small spheres or cubes), this integral is meaningful in the intuitive sense without needing to do anything special.

In the case of ρ constant, we are simply tracking the volume of the chunk. Conservation of mass therefore requires that

$$\frac{\partial}{\partial t}\left\{ \int_{\mathbf{r} \in R(t)} dV \right\} = 0. \tag{10}$$

Our particles within the chunk have all translated an amount $\mathbf{u}(\mathbf{r}_0)\delta t$, so (in a somewhat clumsy notation) our new offset is

$$\delta\mathbf{r}(t_0 + \delta t) \approx \delta\mathbf{r}_0 + D(\mathbf{r}_0)\delta\mathbf{r}_0\delta t + \frac{1}{2}\boldsymbol{\omega} \times \delta\mathbf{r}_0\delta t \tag{11}$$

with a rotation term (no volume change, but we will still consider it as though it could change volumes for completeness) alongside the deformation from D. The transformation looks like $I + G\delta t$, and we want to know what $\det(I + G\delta t)$ is (since we are interested in volume changes). Fortunately, Jacobi has done all the hard work for us:

$$\frac{d}{dt}\det(A(t)) = \mathrm{tr}\left(\mathrm{adj}(A(t))\frac{dA(t)}{dt}\right) \tag{12}$$

and we have the simple job of plugging in $dA/dt = G$ and $A(t_0) = I$ (hence $\mathrm{adj}(A(t_0)) = I$), finding that the rate of change of volume is simply the trace of G (which is contained in D, justifying the claim that the rotation could not change volumes), therefore mass is conserved if $\mathrm{tr}(G) = \mathrm{tr}(D) = \nabla \cdot \mathbf{u} = 0$, and we can all go home happy[1].

And so we reach the 'whatever's left over' part. In continuum mechanics, where tensors are often used, the quantities d_{ij} and ω_{ij} are often referred to as the 'rate of strain' and 'spin/rotation' tensors, corresponding to our D and W. We're yet to do anything particularly interesting with the off-diagonal elements of D, beyond noting that they represent the correlation between axial deformations (we have essentially shown the symmetry of shear). Let's fix that by putting all the expansion into a single, uniform term:

$$\begin{aligned} D &= \left(\frac{1}{3}\nabla \cdot \mathbf{u}\right) I + \left(D - \left(\frac{1}{3}\nabla \cdot \mathbf{u}\right) I\right) \\ &= S + D'. \end{aligned} \tag{13}$$

This expansion term is sometimes called the 'spherical' part, or 'rate of expansion' tensor, whilst adjusting our leftovers into D' yields a deformation at constant volume, sometimes called the 'deviatoric' part or 'rate of shear' tensor. Note the factor of $1/3$ required to ensure that $\mathrm{tr}(S) = \nabla \cdot \mathbf{u}$ and $\mathrm{tr}(D') = 0$: in n dimensions we apply $1/n$ of the divergence to each diagonal element to create a uniform expansion.

So, putting it all together, what does the behaviour of our fluid look like around a point? With

$$G = S + W + D', \tag{14}$$

we have:

$$\mathbf{u}(\mathbf{r}_0 + \delta\mathbf{r}) \approx \mathbf{u}(\mathbf{r}_0) + \left(\frac{1}{3}\nabla \cdot \mathbf{u}\right)\delta\mathbf{r} + \frac{1}{2}\boldsymbol{\omega} \times \delta\mathbf{r} + D'\delta\mathbf{r}, \tag{15}$$

representing:

- $\mathbf{u}(\mathbf{r}_0)$: moving with the flow;
- $\left(\frac{1}{3}\nabla \cdot \mathbf{u}\right)\delta\mathbf{r}$: uniformly expanding away from $\mathbf{r}_0$;
- $\frac{1}{2}\boldsymbol{\omega} \times \delta\mathbf{r}$: rotating around $\mathbf{r}_0$ with angular velocity $\frac{1}{2}|\boldsymbol{\omega}|$ and axis $\hat{\boldsymbol{\omega}}$; and
- $D'\delta\mathbf{r}$: shearing at $\mathbf{r}_0$.

[1] In the case of ρ not constant, we gain a material derivative term (after all, we are literally following the blob) in dm/dt of $\int(\partial\rho/\partial t + \mathbf{u} \cdot \nabla\rho)dV$, and we can merge the $\mathbf{u} \cdot \nabla\rho$ into the deformation term for the more traditional-looking form $\partial\rho/\partial t + \nabla \cdot (\rho\mathbf{u}) = 0$. We still go home satisfied that maths is not yet broken.